An Approximate Analytical Approach of Water Transport in an Unsaturated Porous Medium by Modified Variational Iteration Method

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One of the most familiar equations for examining the properties of infiltration in unsaturated regions of soil like a porous media is known as Richards' equation. The main aim of this paper is to illustrate the behaviour of the water transport problem in an unsaturated soil. The Variational Iteration Method and the Modified Variational Iteration Method have been used to determine an approximate analytical solution for Richards Equation at different cases. A comparison of the solution has been made with the exact solution, Finite Difference Method, and Elzaki Adomain Decomposition Method (EADM) solutions. For the numerical and graphical representation, MAPLE and MATLAB software are used.

Keywords: Richard's Equation, Variational Iteration Method (VIM), Modified Variational Iteration Method (MVIM), Elzaki Adomain Decomposition Method (EADM), Infiltration.

1 Introduction

Various kinds of phenomena occur in the scientific and engineering fields that are usually represented using ordinary differential equations or partial differential equations. Many scientists have worked hard to model water penetration in unsaturated soil [16, 19, 23, and 28]. For non-linear equations, there are various

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different types of numerical [11, 23 and 25] and analytical methods [5, 20 and 30] are available. Although there are few analytical ways to simulate infiltration, there are a number of different numerical studies that try to simulate this physical process. The MVIM and VIM are two latest methods for solving non-linear partial differential or mathematical equations that can provide a convenient analytical solution. Engineers, non-specialists and several others are using the MVIM to convey non-linear problems while this provides an estimated closed form of solution. MVIM can manage highly non-linear differential equations without using small variables. However, according to the Variational iteration theory, the solution algorithm is efficient, and only a few iterations are required to reach good precision. [24]Richard's equation is generally a partial differential equation describing water transport in unsaturated soils. Richards equation is modelled by various numerical methods [11, 13 and 25]. Infiltration through unsaturated soil is widely characterized by the accumulation of precipitation or moisture surface processes. Richards developed a continuum mechanics-based governing equation for water flow in soil. The continuity equation was combined with Darcy's law as a momentum equation in this model. The mixed formulation, the h - based formulation, and the θ -based formulation are the three main kinds of occurring equations discussed in the paper [8, 12 and 27], where h is the weight-based pressure potential and θ is the unsaturated soil moisture content. In a moisture less soil, the one-dimensional form of Richards Equation can be computed with Darcy's law and the continuity equation, as shown below.

$$q = -K\left(\frac{\partial h}{\partial z} + 1\right) \tag{74.1}$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial q}{\partial z} \tag{74.2}$$

K represents unsaturated hydraulic conductivity, q represents flux density, and t represent time. A mixed form of Richards' equation is constructed by inserting equation (1) in (2):

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K\left(\frac{\partial h}{\partial z} + 1\right)\right] \tag{74.3}$$

This equation is known as Richards equation which governs water movement in the soil [26]. Two independent variables in equation (3) are soil water content (θ) and pore water pressure head (h). To discuss an interrelation between hydraulic conductivity, saturation, and pressure. There are native relations that are demanded to gain the result of the equation. Assuming that the differential water capacity is used to remove either θ or h. So, for that soil water retention curve is defined as:

$$C(h) = \frac{\partial \theta}{\partial h} \tag{74.4}$$

The equation (3) is inserted in equation (4) is used to define the h-based expression of Richard's equation.

$$C(h) * \frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) + \frac{\partial K}{\partial z} \tag{74.5}$$

The ratio of hydraulic conductivity (K) to differential water capacity (C) is termed as pore water diffusivity (D). The θ - based expression of Richard's equation is followed as:

$$D = \frac{K}{C} = K\frac{\partial h}{\partial \theta} \tag{74.6}$$

Both D and K are pore water diffusivity and hydraulic conductivity is both modified by (moisture content). Richard's equation is formed by combining the equations (6) and (3):

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial \theta}{\partial z}\right) + \frac{\partial K}{\partial z} \tag{74.7}$$

D and K are the dependent parameters and both are difficult to evaluate. So, there are different kinds of models that have been developed to evaluate these parameters. The most useful models are the exponential models [12], the Van Genuchten model [27], and Brook's and Corey's model [8]. The functional model in Van Genuchten's model compares experimental data, however, it is highly complex. Despite the fact that Brooks and Corey's model [8] is clearly defined or more exact value that is associated with greater pore size. Some relations are presented to define D and K according to Brook's and Corey's [8].

$$D(\theta) = \frac{K_s}{\alpha\lambda\left(\theta_s - \theta_r\right)}\left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{2+\frac{1}{\lambda}} \tag{74.8}$$

$$K(\theta) = K_s\left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{2+\frac{1}{\lambda}} \tag{74.9}$$

where K_s is saturated hydraulic conductivity, θ_s and θ_r are saturated soil moisture content and residual water content respectively. α and λ are parameters that were determined by the experiment. The pore size distribution index [8] is used to determine Brooks and Corey. Brooks-Corey model soils can be simplified to the following equations by performing some additional considerations [8 and 10].

$$D(\theta) = D_0(m+1)\theta^m, m \geq 0; \tag{74.10}$$

$$K(\theta) = K_0\theta^k, k \geq 1 \tag{74.11}$$

K_0 is the constant. The soil parameters such as pore size distribution and particle size are indicated by D_0 and k.

θ is applied between 0 to 1 for the presentation of D and K, and diffusivity is reduced such that $\forall m$, $\int_0^1 D(\theta)d\theta = 0$ [21] . Various numerical and analytical solutions to Richard's equation have been examined based on Brook's and Corey's depiction of D and K. Putting $m = 0$ and $k = 2$ in equations (10) and (11) gives the standard forms of Burger's equation. Burgers' equation has also been investigated by several researchers [6,7 and 30]. Instead of time and depth, another variable that could be a linear mixture of them is discovered using the travelling wave technique [21]. To solve these transform equations, a tangent-hyperbolic function is commonly used. As a result, the θ - based Richards' equation in order of (m, 1) in [21]:

$$\theta_t + \alpha\theta^m\theta_z - \theta_z z = 0 \tag{74.12}$$

The Equation (12) is an exact solution of Richard's equation

$$\theta(z,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}tanh(B_1(z - B_2t))\right)^{\frac{1}{m}} \tag{74.13}$$

where $B_1 = -\frac{\alpha m + m|\alpha|}{4(1+m)}\gamma(m \neq 0), B_2 = \frac{\gamma\alpha}{(1+m)}$. Here α and γ are arbitrary constant and taken as 1 in this paper [14]. Considering $t = 0$ then gives initial conditions. The non-linear based Richard's equation was examined in this work.

VIM and MVIM were introduced in the following sections, and also used to the Richards Equation to have an approximate analytical solution for equation (12), with comparisons to the exact solution and the Elzaki Adomain Decomposition Method [28] have been shown.

2 Methods

The approach of He's Variational Iteration Method [2, 14, 20, 21, 24 and 29] can be described by assuming the non-linear partial differential equation.

$$L\left(U(x,t)\right) + N\left(U(x,t)\right) + R\left(U(x,t)\right) = f(x) \tag{74.14}$$

U(x,0)=g(x)

where $f(x)$ is an inhomogeneous term, $L(x)$ is a linear term, $N(x)$ is a non-linear term and R is a linear operator which has partial derivatives with respect to x. In accordance with the method, we can set up a correction functional for the equation (14) is as follow:

$$U_{n+1}(x) = U_n(x) + \int_0^x \lambda(x)\left[L(U_n(x)) + R(\widetilde{U_n}(x)) + N(\widetilde{U_n}(x)) - f(x)\right] d\tau \tag{74.15}$$

while λ is a Lagrange multiplier [19] that the VIM can be recognized. [4,15,17,19,22 and 29]$U_0(x)$ is an initial approximation with possible unknowns, U_n denotes the nth approximation, and $\widetilde{U_n}$is considered as a restriction variation i.e. $\delta \widetilde{U_n} = 0$. The Lagrange Multiplier and the initial guess U_0 may quickly calculate the successive approximation $U_{n+1}, n \geq 0$ of the solution U,and hence the solution is $U = \lim_{n\to+\infty} U_n$. Equation (15) can be solved iteratively using $U_0(x,t) = g(x)$.

The Modified Variational Iteration Method [1, 2, 3] considers the same procedure of Variational iteration Method, but instead of using correction functional equation (15) we will use this iterative formula:

$$U_{n+1}(x) = U_n(x) + \int_0^x \lambda(x)\left[R(U_n - U_{n-1}) + (G_n - G_{n-1}) - f(x)\right] d\tau \tag{74.16}$$

where $U_{-1} = 0$, $U_0 = f(x)$ and $G_n(x,t)$ is obtain from

$$N(U_n(x,t)) = G_n(x,t) + O(t^{n+1}) \tag{74.17}$$

For obtaining an approximate solution Equation (14) can be solved iteratively and get the form

$$U(x,t) \cong U_n(x,t)$$

where n final iteration step.

3 Results

In this part, we had applied MVIM and VIM to get the results for Richards Equation. For the purpose of convenience, the two different cases of m in equation (12) are considered.

3.1 Case 1: if $m = 1$

The proposed method is applied to solve Richard's Equation at m=1. so the equation (12) can be written as:

$$\frac{\partial \theta}{\partial t} + \theta \frac{\partial \theta}{\partial z} - \frac{\partial^2 \theta}{\partial z^2} = 0 \tag{74.18}$$

with initial condition

$$\theta(z, 0) = \left(\frac{1}{2} + \frac{1}{2} \tanh \left(-\frac{z}{2} \right) \right) \tag{74.19}$$

Applying VIM in the above equation,

$$\theta_{n+1}(x) = \theta_n(x) + \int_0^x \lambda(x) \left[\frac{\partial \theta_n}{\partial t} + \theta_n \frac{\partial \theta_n}{\partial z} - \frac{\partial^2 \theta_n}{\partial z^2} \right] d\tau \tag{74.20}$$

Its stationary condition will come as follow:

$$\lambda^{'}(\tau) = 0$$

$$1 + \lambda(\tau)|_{\tau = t} = 0$$

The Lagrange Multiplier can be recognized as the $\lambda = -1$. Consequently, the correction functional will be revised and the initial condition (19) will be treated as the initial estimate.

$$\theta_0(z, t) = \left(\frac{1}{2} + \frac{1}{2} \tanh \left(-\frac{z}{4} \right) \right)$$

After using the initial approximation in equation (19) we can get the approximate solution:

$$\theta_1 = \frac{1}{2} - \frac{1}{2} \tanh \left(\frac{z}{4} \right) + \frac{1}{16} t sech \left(\frac{z}{4} \right)^2$$

$$\theta_2 = \frac{1}{2} - \frac{1}{2} \tanh \left(\frac{z}{4} \right) + \frac{1}{16} t sech \left(\frac{z}{4} \right)^2 + \frac{t}{64} \tanh \left(\frac{z}{2} \right) sech \left(\frac{z}{4} \right)^2 \Big[1 + t \frac{t}{16} + \frac{t}{2}$$

$\tanh\left(\frac{z}{4}\right) + \frac{t}{2} tanh \left(\frac{z}{4}\right)$

and so on.

The approximate solution is obtained as:

$$\theta = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{z}{4}\right) + \frac{1}{16}tsech\left(\frac{z}{4}\right)^2 + \frac{t}{64}\tanh\left(\frac{z}{4}\right)sech\left(\frac{z}{4}\right)^2\left[1 + t\frac{t}{16} + \frac{t}{2}\right.$$

$\tanh\left(\frac{z}{4}\right) + \frac{t}{2}tanh\left(\frac{z}{4}\right) + ...$

Applying MVIM with the help of equation (16) in Richards's equation (12). where $\theta_{-1} = 0, \theta_0 = 0, \theta_1(x) = \theta_0 - \int_0^x [R(\theta_0 - \theta_{-1}) + (G_0 - G_{-1}) - f(x)]\, d\tau$ and from the equation (17) $G_n(x,t)$ is obtained as:

$$(\theta_n(x,t))\,(\theta_n(x,t))_x = G_n(x,t) + O(t^{n+1})$$

After applying the modified formula equation(16), at the end it will cancelled out all the additional recurring calculations and terms. After applying MVIM, the following solutions are obtained.

$$\theta_0(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{z}{4}\right)\right)$$

$$\theta_1(z,t) = \theta_0 + \frac{1}{16}tsech\left(\frac{z}{4}\right)^2$$

$$\theta_2(z,t) = \theta_1 + \frac{t^2}{1024}\tanh\left(\frac{z}{4}\right)sech\left(\frac{z}{4}\right)^2 + \frac{t^2}{128}\tanh\left(\frac{z}{4}\right)^2 sech\left(\frac{z}{4}\right)^2$$

and so on.

MVIM makes a calculation faster and makes the solution in a simplified form. Tables 1, 2 and 3 display the comparison of the solution obtained by MVIM, VIM, EADM and Exact solution with z=0,1,2,3,4 and 5 for t=1,3 and 5 . Furthermore, the absolute error between the solution and the exact solution generated by the proposed method MVIM could be seen in the tables below. Table 4 shows the comparison of solution with numerical method Finite Difference Method (FDM) and absolute error has been showed. At t=3, m=1, Figure 1 presents a comparison between the results obtained by MVIM, VIM, EADM, and the exact solution. Figure 2 depicts three dimensional behaviour of an approximate solution for the proposed method (m=1).

3.2 Case 2: if $m = 2$

The proposed method is applied to solve Richard's Equation at m=2. So the equation (12) can be written as:

Table 1: For m=1 and t=1 numerical comparison of solution with different methods and absolute error.

z	MVIM	VIM	EADM	Exact	Error = |Exact-VIM|	Error = |Exact-MVIM|
0	0.5625	0.5625	0.5625	0.562177	0.000323	0.000323
1	0.4369	0.44142	0.43809	0.437823	0.003597	0.000923
2	0.3197	0.32799	0.320934	0.320821	0.007169	0.001121
3	0.2218	0.2314945	0.222672	0.2227	0.0087945	0.0009
4	0.1475	0.156289	0.14795	0.148047	0.008242	0.000547
5	0.0950	0.1018953	0.0952425	0.0953495	0.0066528	0.0003495

Table 2: For m=1 and t=3 numerical comparison of solution with different methods and absolute error.

z	MVIM	VIM	EADM	Exact	Error = |Exact-VIM|	Error = |Exact-MVIM|
0	0.6875	0.6875	0.6875	0.679179	0.008321	0.008321
1	0.55766	0.573135	0.569981	0.562177	0.010958	0.004517
2	0.430721	0.45563017	0.441954	0.437823	0.0178072	0.007102
3	0.3131944	0.34223777	0.320928	0.320821	0.0214168	0.0076266
4	0.21625566	0.24266809	0.220438	0.2227	0.01996809	0.00644434
5	0.14321004	0.1638188	0.145161	0.148047	0.0157718	0.00483696

Table 3: For m=1 and t=5 numerical comparison of solution with different methods and absolute error.

z	MVIM	VIM	EADM	Exact	Error = |Exact-VIM|	Error = |Exact-MVIM|
0	0.8125	0.8125	0.8125	0.7773	0.0347	0.0347
1	0.687592	0.7099839	0.716261	0.679179	0.0308049	0.008413
2	0.5544866	0.5960	0.585689	0.562177	0.033823	0.0076904
3	0.4213869	0.4698	0.442867	0.437823	0.031977	0.0164361
4	0.3013017	0.345321	0.312916	0.320821	0.0245	0.0195193
5	0.204527	0.238875	0.2099471	0.2227	0.016175	0.018173

Table 4: For m=1 comparison of solution with numerical method FDM and absolute error.

	FDM			Absolute Error		
z	t=1	t=3	t=5	t=1	t=3	t=5
0	0.5000	0.5000	0.5000	0.062177	0.179179	0.2773
1	0.3921	0.4023	0.4100	0.044723	0.159877	0.269179
2	0.2892	0.3053	0.3180	0.031621	0.132523	0.244177
3	0.2024	0.2189	0.2317	0.0203	0.101921	0.206123
4	0.1343	0.1448	0.1526	0.013747	0.0779	0.168221
5	0.0759	0.0751	0.0759	0.0194495	0.072947	0.1468

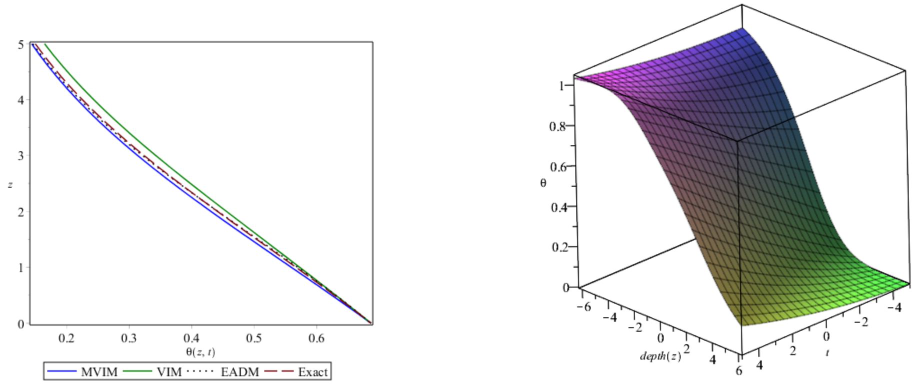

Figure 2: Three dimensional behaviour of analytical approximate solution at m=1.

$$\frac{\partial \theta}{\partial t} + \theta^2 \frac{\partial \theta}{\partial z} - \frac{\partial^2 \theta}{\partial z^2} = 0 \tag{74.21}$$

with initial condition

$$\theta(z,0) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{z}{3}\right)\right)^{0.5} \tag{74.22}$$

Applying VIM in the above equation,

$$\theta_{n+1}(x) = \theta_n(x) + \int_0^x \lambda(x)\left[\frac{\partial \theta_n}{\partial t} + \theta_n^2 \frac{\partial \theta_n}{\partial z} - \frac{\partial^2 \theta_n}{\partial z^2}\right] d\tau \tag{74.23}$$

The Lagrange Multiplier can be recognized as the $\lambda = -1$ and considering an initial condition (22) as an initial approximation

$$\theta_0(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{z}{3}\right)\right)^{0.5}$$

After using the initial approximation in equation we can get the approximate solution is obtained as:

$$\theta(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{z}{3}\right)\right)^{0.5} - \frac{tsech\left(\frac{z}{3}\right)^4}{144\left(0.5 - 0.5tanh\left(\frac{z}{3}\right)\right)^{\frac{3}{2}}}$$
$$+\frac{tsech\left(\frac{z}{3}\right)^2}{24\left(0.5 - 0.5tanh\left(\frac{z}{3}\right)\right)^{0.5}} + \frac{tsech\left(\frac{z}{3}\right)^2 tanh\left(\frac{z}{3}\right)}{24\left(0.5 - 0.5tanh\left(\frac{z}{3}\right)\right)^{0.5}} + \ldots$$

Applying MVIM with the help of equation (16) in Richards's equation (12). where $\theta_{-1} = 0, \theta_0 = 0, \theta_1(x) = \theta_0 - \int_0^x [R(\theta_0 - \theta_{-1}) + (G_0 - G_{-1}) - f(x)]\, d\tau$ and from the equation (17) $G_n(x,t)$ is obtained as:

$$(\theta_n(x,t))\,(\theta_n(x,t))_x = G_n(x,t) + O(t^{n+1})$$

After applying the modified formula (16), at the end it will cancelled out all the additional recurring calculations and terms. After applying MVIM, the following solutions are obtained.

$$\theta_0(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{z}{3}\right)\right)^{0.5}$$

$$\theta_1(z,t) = \theta_0 + \frac{tsech\left(\frac{z}{3}\right)^4}{24\left(0.5 - 0.5tanh\left(\frac{z}{3}\right)\right)^{0.5}}\left[1 - \frac{sech\left(\frac{z}{3}\right)^2}{6\left(0.5 - 0.5tanh\left(\frac{z}{3}\right)\right)} + \frac{tanh\left(\frac{z}{3}\right) sech\left(\frac{z}{3}\right)^2}{3\left(0.5 - 0.5tanh\left(\frac{z}{3}\right)\right)}\right]$$

and so on.

MVIM makes a calculation faster and makes the solution in a simplified form. Tables 5,6 and 7 display the comparison of the solution obtained by MVIM, VIM, EADM and Exact solution with z=0,1,2,3,4 and 5 for t=1,3 and 5 . In addition, the error between the precise solution and the solution derived using the proposed technique MVIM and VIM is shown in the tables below. Table 8 shows the comparison of solution with numerical method FDM and absolute error at t=1,2,3 and z=0,1,2,3,4 and 5 has been showed. At t=3, m=2, Figures represents the comparisons of results of MVIM, VIM, EADM, and the exact solution. Figure 4 depicts the three dimensional behaviour of an approximate solution for the proposed method (m=2).

Table 5: For m=2 and t=1 numerical comparison of solution with different methods and absolute error.

z	MVIM	VIM	EADM	Exact	Error = |Exact-VIM|	Error = |Exact-MVIM|
0	0.74639	0.7463904	0.700287	0.745203	0.0011874	0.001187
1	0.62521	0.625208	0.58141	0.625046	0.000162	0.000164
2	0.4969	0.4968988	0.465194	0.497658	0.0007592	0.000758
3	0.370469	0.37046876	0.361456	0.380234	0.00976524	0.009765
4	0.2813717	0.28137167	0.273559	0.28257	0.00119833	0.0011983
5	0.2055054	0.20550543	0.202672	0.206522	0.0010165	0.0010166

Table 6: For m=2 and t=3 numerical comparison of solution with different methods and absolute error.

z	MVIM	VIM	EADM	Exact	Error = |Exact-MVIM|	Error = |Exact-VIM|
0	0.82497	0.8249679	0.704652	0.812869	0.012101	0.0120989
1	0.710731	0.7107313	0.576126	0.707107	0.003624	0.0036243
2	0.577223	0.5772227	0.461495	0.582446	0.005223	0.0052233
3	0.446625	0.4466251	0.376334	0.456737	0.010112	0.0101119
4	0.334334	0.33433399	0.303302	0.345258	0.010924	0.01092401
5	0.245328	0.24532795	0.235861	0.254891	0.009563	0.00956305

Table 7: For m=2 and t=5 numerical comparison of solution with different methods and absolute error.

z	MVIM	VIM	EADM	Exact	Error = |Exact-VIM|	Error = |Exact-MVIM|
0	0.903525	0.90352533	0.733023	0.867373	0.036152	0.03615233
1	0.7962547	0.796254662	0.566559	0.780588	0.0156667	0.01566662
2	0.6575466	0.6575466	0.430311	0.666837	0.0092904	0.0092904
3	0.5142033	0.5142033	0.367854	0.539758	0.0255547	0.0255547
4	0.3872963	0.38729633	0.322919	0.417475	0.0301787	0.03017867
5	0.2851505	0.28515047	0.267762	0.312686	0.0275355	0.02753553

4 Discussion

MVIM and VIM are effectively implemented to the Richards equation for obtaining the approximate analytical solution in this work. To substantiate the approximation solution by VIM and MVIM, for both the cases at m = 1 and 2 are compared with the exact solution and EADM [28]. Table 4 and 8 shows the comparison with the numerical method (FDM) at m=1 and 2, t=1, 3, 5. As the MVIM solution steps in the Richards equation are compared with the EADM solution steps in [28], from figures show that it is clear that MVIM is more efficient. Adomain polynomials with six non-linear terms must be computed in EADM. These polynomials do not need to be calculated in MVIM. When moving out from the transition zone, all of figures 1-4 show that VIM and MVIM produce more accurate results. Maple and MATLAB software has been used for graphical and numerical representation. The results reported in this study, however, suggest that the VIM and its expansion can often be implemented in several non-linear partial differential equations. The approximate solution is converged while MVIM minimizes the repeating of previous computations and eliminates out unnecessary terms.

Table 8: For m=2 comparison of solution with numerical method FDM and absolute error.

	FDM			Absolute Error		
z	t=1	t=3	t=5	t=1	t=3	t=5
0	0.7071	0.7071	0.7071	0.038103	0.105769	0.160273
1	0.5901	0.5968	0.6024	0.034946	0.110307	0.178188
2	0.4707	0.4826	0.4923	0.026958	0.099846	0.174537
3	0.3617	0.3751	0.3853	0.018534	0.081637	0.154458
4	0.2690	0.2778	0.2842	0.01357	0.067458	0.133275
5	0.1856	0.1856	0.1856	0.020922	0.069291	0.127086

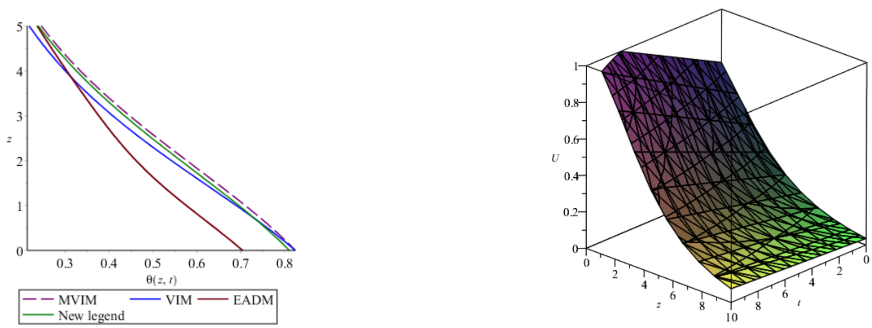

Figure 4: Three dimensional behavior of analytical approximate solution at m=2.

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