A Review on Curvelets and its Applications

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Wavelets have a large impact on image and signal processing, but it is observed that they fail to represent an object like linear or curvilinear structure which is highly anistropic. To alleviate this problem,curvelets have been introduced by Candes and Demanent. Curvelets have shown a great interest in the field of image and signal processing over the past few years. The beauty of curvelets over wavelets is that it can be constructed over general manifolds. In this paper, we represent the review on curvelets transform, including its history from wavelets, and so forth. Moreover, this paper will also demonstrate the numerous applications of curvelets such as seismic data recovery, X-Ray computed tomography, interferometry images, time-frequency analysis, processing of MRI for local image enhancement and noise suppression of receivers function, which would be fruitful for discovering the real life applications and problems that can be solved with the help of curvelets.

Keywords: Curvelet, Wavelet, Multiscale geometric analysis, Seismic data recovery, Interferometricimage.

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1 Introduction

Numerous natural images and signal sexhibitline like edgesi.e., they are intermittent along the curve singularities. The digital images are the two- dimensional matrix and we have to adjust these matrices to get the limpid image for the image process. Albeit, wavelets are competent in capturing horizontal, vertical, and diagonal directions i.e., such edges, but it proves to be inadequateforcapturing details along the curvature. Curvelet transform is a multiscale geometric transform that is more efficient than regular wavelets in representing edges and curve singularities. The frequency domain is where the curvelets transform is typically used. Curvelets are effective in modelling hyperbolic differential equations with highly general linear symmetric systems. [2]. The curvelet transform provides them with smooth images where the edges are sporadic and the curvelet transform is based on combining several ideas of ridgelets, multiscale, and band filtering [3].

Another point of concern about curvelets is that their decomposition is rough twice the approximation rate of wavelet decompositions and is about four times of the Fourier decomposition[4]. Curvelets are a new multiscale representation for bivariate functions that were first developed in relation to the central problem in approximation theory and statistical estimation [5]. Wavelets, unlike curvelets, are challenged by small features such as lens eyes since they do not restore long edges with great fidelity. Curvelets have the ability to transform their field of competence, and this complementarity has a lot of potential [6]. Althoughthetheoretical concept of wavelets is quite easy, its implementation in the practical world is strenuous.

This papers broadly consist of 6 Sections. In Section 2 we discussed about the transformation from classical wavelets to curvelets. In Section3, the main aim of curvelets and how we define curvelets has been shown. In section4, literature review of the paper has been discussed which is very helpful for the beginners. In Section 5, conclusion and future work of the paper is given.

2 From Classical Wavelet to Curvelet

Wavelet transform and fourier transform reigned supreme before curvelets. We express signals in terms of sinusoids in the Fourier transform, which produces a signal that is only localised in the frequency domain and does not produce a signal in the time domain, implying that there is a difference in time but the same frequency representation. Only the Fourier transform reveals which frequency components are present in a signal. The Fourier transform cannot determine where a time-frequency component occurs, although in most circumstances, time-frequency representation is required.

For both stationary and non-stationary signals, Fourier is a good choice. The Fourier transform, on the other hand, is thought to have a number of problems, such as when the signal is discontinuous.

- Fourier transform produces a large coefficient with large magnitude and many more.
- After Fourier transformi.e., to overcome the limitations of Fourier Transform we discovered two methods i.e., wavelet transform and short-time Fourier analysis.

Now Denis Gabor in 1946 developed a technique based on Fourier transform [1]and using windowing but in these wide windowsare more appropriate at low frequency and narrow windows are more appropriate at high frequency. From this, we switch to the concepts of wavelets in which they can be localized in both the time and frequency domain. In wavelets a function that waves above and below the X-axis with the following properties are: varying frequency, limited duration, zero average value.

- There are many advantages of using wavelets i.e., it provides a way for analyzing waveforms in both frequency and duration.

- It represents a function that has discontinuity and sharp peaks.

It is possible to accurately deconstruct and recreate finite non-periodic and non-stationary signals. It enables for more efficient signal storage than the Fourier transform. Wavelets, on the other hand, are good at capturing horizontal, vertical, and diagonal directions, i.e., edges. But it's not good if we need to record details along the curve. The curvelet transform is a multi-scale geometric transform that can be used to represent edges and curves. Traditional wavelets are inefficient compared to singularities. The downside of wavelets is that they have weak directionality. The complex wavelets transform, on the other hand, is on its way to improving directional selectivity. However, creating a complicated wavelet with flawless reconstruction properties and good filter characteristics is difficult. Ridgelet transform is an anisotropic geometric wavelet transform proposed by Candes and Donoho in the year 1999 [2].

Ridgelet transform based curvelet transfrom was introduced in 2000 by Candes and Donoho and was dubbed first generation curvelets transform [3]. The same author later introduced the second generation curvelet transform [4]. It's a powerful tool for image processing, seismic data exploration, fluid physics, and solving partial differential equations, among other things. Although the discrete curvelet transform is very efficient in representing curves and edges, the second generation Curvelet transform has two major flaws:

- It is not optimal for sparse approximation of curve features beyond C^2 – singularities
- It is highly redundant.

After that, the curvelet transform was introduced, which was faster and less redundant. The fast discrete curvelet transform (FDCT) is a recently developed curvelet transform that may be implemented in two ways: unevenly spaced fast Fourier transform and wrapping function [4]. Because the curvelet transform is commonly performed in the frequency domain, the following equation can also be stated in that domain.

2.1 What Are Curvelets

Curvelets are very upcoming topics in today's digital world. All the digital appliances which work on image and signal processing needs to get enhanced day by day and capture the signals at every point whether it is discontinuous at that particular point or not. Curvelets are much better than wavelets because it has many places it works smoothly where the wavelets ideals are not that appropriate.

Curvelets are much smoother at the edges and at the bounded places that's why we say that curvelets are an optimally sparse representation of the objects at their edges.Let FM be the m-term in approximation theory. Curvelet approximation to an object f(x1, x2) L2 (R2) [7], corresponding to the m biggest coefficients in the curvelet series. Then there's the increased sparsity.

The approximation error obeys if the object f is singular along a general smooth C2 curve but otherwise smooth.

$$f - fm^2$$

$$L^2 \leq C \cdot (\log m) \quad (1)$$

$$3 \cdot m-2$$

Curvelets are also useful for analysing the partial differential equation, which we refer to as anoptimally sparse wave propagation model. One of the most essential aspects of curvelets is that their coefficients rapidly decline for continuous functions, whereas for functions with one of their derivatives that is discontinuous, the curvelet coefficient decreases moderately close to the discontinuity and rapidly far away. This characteristic aids in locating the point in the numerical partial differential equation [8] where the disturbance occurs.

A few years back curvelets were very difficult to use but after continue working on themcurvelets have been redesigned so that it becomes easy to work on them. And the most interesting part after those innovative algorithm strategies has been suggested by new mathematical architects and they also provide opportunities for improving the earlier implementations [9].

3 Literature Review

Table 1. Literature review

Year	Author Name	Paper Name	Key Findings
1999	D.L. Donoho and M.N. Duncan	Digital Curvelet transform: strategy, implementations, and explanations	This paper shows that howcurvelets has been used practically i.e., practical use of curvelets and also tells that in today's world where digitalization is at its peak howcurvelets has been used for getting things more perfect and accurate. It also discusses the relationofcurvelets and ridgelets and also discuss the digital Ridgelet transform.
2000	E.J. Candes and D.L.Donoho	Curvelets: A surprisingly effective and non-adaptive representation of objects with edges.	This paper discusses about the information to use curvelets for nonadaptive representation for objects and edges. This paper shows the basic issue of m-term representation, and also elaborate about the construction of efficient adaptive representation and curvelets frames.
2003	E.J.Candes and D.L. Donoho	Gray and color image contrast enhancement by curvelet transform	This paper works on the contrast enhancement of the images using curvelet transforms and Ridgelets transform it also shows that when wavelets has been used then the enhancement of the images at its edges was not that good as after using curvelet but for getting the noise-free image both curvelets and wavelets show the same features.
2003	E.J.Candes and L.Demanet	Curvelets and Fourier transform operator	This paper works upon that the Fourier and wavelets do not work on sparse representation of FIOs and it explains what is curvelets

			and how it works and the importance of parabolic scaling.
2003	J.L. Starcka , M.K.Nguyenb and F.Murtaghc	Wavelets and Curvelets for image deconvolution	This paper discusses about wavelets and the transformation from wavelets to curvelets and also discusses the filtering and deconvolution of the images and many more.
2003	E.J.Candes and D.L.Donoho	New tight frames of curvelets and optimal representations of an object with Piecewise C2 singularities	This paper works from the new tight frames of curvelets and also it helps in the optimal representation of objects in this discussion aboutvarious types of new and tight frames of curvelets and also about the functions of C^2 away from C^2 edges and also tells about the Fourier analysis of edges fragments.
2007	L.Demaneta and L.Yingb	Curvelets and wave atoms for mirror-extended images	This paper gives information about mirror extended curvelets and how to work on mirror extended wave atoms and it shows how to work on Neumann boundary conditions.
2008	B.Zhang, J.M.Fadili and J.L.Starck	Wavelets Ridgelets and curvelets for Poisson noise removal	This paper talks about curvelets and how they can be solved through filtered poison processed helps to get know about how multiscale variance stabilizing transform and the poison intensity estimation. This paper shows how curvelets are related toridgelets and wavelets and also discuess about how curvelets are better options nowdays.
2008	J.L.Starcka , M. K. Nguyenb and F.Murtaghc	Wavelets and curvelets for image deconvolution	This paper talks about wavelets and why to switch from wavelets to curvelets and also discusses the filtering and deconvolution of the images and many more.
2008	R.Neelamani,A.I.B aumstien and D.G.Gillard	Coherent and noise attenuation using curvelet transforms.	This paper builds on the findings by using curvelet-based noise attenuation to a noisy seismic cubic from a carbonate environment, although the results aren't completely satisfactory. The new advancement will be enabled via Curvelet transform. But this curvelet is only the powerful

			approach for facing the problem of recurring seismic noise attenuation.
2008	S.Arivazhagan,L.Ganesan and T.G.Subashkumar	Texture classification using curvelet statistical and co-occurance features.	This paper works on how texture classification can be done using image processing and by using wavelets it will be good only in point singularities.Curvelet sub bands are more powerful than that of wavelet sub bands.
2011	S.Prasad, P.Kumar and R.C.Tripathi.	Plant leaf species identifications using curvelet transform.	This paper gives us the information to recognise the plant species which is very helpful for the botanical students. In this a new multiresolution and multidividedcurvelet transform is used to get the clear image of the divisions in the leaves and it also helps in recognition of the various plants and species. By usingcurvelets the image quality becomes clear and can get the name of the species as soon possible with support of vector machine.
2014	Mohd.M.Eltoukhy, I.Faye and B.B.Sami.	The comparison of wavelets and curvelets for breast diagnosis in digital program	This paper discusses about one more application of the curvelets i.e., how do we use wavelets as well as curvelets for breast diagnosis in digital program. It also talks aboutmultiresolution analysis for image processing at different level and also about the feature extraction.
2019	D. Sharma, R. K. Singhla and K. Goyal	An Adaptive grid-based curvelet optimized solution of non-linear Schrodinger equation	The dynamically adaptive curvelets technique for solving the Non-liner Schrödinger equation is the subject of this paper. The multiresolution analysis, which is employed in image processing, was also explained. It also mentions that the grid is generated using a curvelet.
2020	D. Sharma, R. K.Singhla and K. goyal	A curvelet method for the numerical solution of partial differential equations	This paper gives us the information about how wavelets are used to solve the partial differential equation and it also explore our knowledge about the curvelet-based numerical methods

			and also the significant characteristics of wavelets i.e., it can be used to design the generalized manifolds. It also shows that the dynamic adapted curvelet approach works are smooth with the local Schrödinger equation.

4 Applications of Curvelets

Curvelets have a wide range of applications so here are some of the applications which has been discussed below:

4.1 Seismic Data Recovery: This Seismic data is used to improve the quality and resolution from incompleteacquired seismic recordings and it has been found that this i.e., that hexagonal jittered sampling performs better than cartians and hexagonal grids.As we know that seismic data and the high dimensional data are very large and they also need a large number of shots for data acquisition and with this, the cost of the hole set leads to get high so we need lesser the number of geophones so that it can be controlled economically and get the better results too. Various results have worked upon blue noise sampling methods to solve the problem[14] of two-dimensional seismic acquisition design[10].Various results and researches are pending on the challenges of using seismic data processing in a hard rock environment. Seismic data is being polluted by the noisy background of various origins and makes the recorded signals do notexhibit coherent nature and it will be mapped using DCT with smalls coefficients. Further it has been found that there isanalgorithm that proves that by using two-dimensional curvelet transform enhancement can be done in 3-dpost-stack seismic data [11].

4.2 X-Ray Computed Tomography: Reconstructing X-Ray computed tomography is verychallenging problem in the field of mathematics.Researchershave worked on it and they have found that evolutions of the methods are performed on the numerical phantom data set which is specially crafted.They also worked on the evaluation of two different commercial scanners with having different noise characteristics and in result it has been found that characteristics strength is available in curvelet sparse regularization[12].Hence it has been found that reconstruction quality can be improved using curvelet.

4.3 MRI Processing For Image Enhancement: Sometimes magnetic resonance does not provide us with the clarified image of bones like structure i.e., that images are blurred at the edges or provide us with noises which are very difficult to identify accurately the bone structure. After that, it has been found that MRI processing feature is effective for this.Various image processing methods have found that curveletstransform is an effective tool for extraction linear features.It has been found that there is a large difference in curveletcoefficient at different pins in boundaries rather than that of smooth edges[13].

4.4 Brain Tumor Detection System Using Gradient-Based Watershed: There is a methodology called GWMAC-CT (gradient-based watershed marked active contours and curvelet transform) that helps to detect brain tumours in magnetic resonance images (MRI).There are various experiments that are carried out from the TRIM NHL protein Brain tumor(Brat) database to set brain MRIimages.And now recently there are various CAD techniques that help to detect brain tumours in magnetic resonance images (MRI). Despite all of the studies, there is still an inability to accurately forecast brain tumours [15]. As a result of the curvelets image processing, we can detect cancers with a high degree of accuracy. The GWMAC-CT technology can be used to detect brain tumours automatically.

4.5 Video Surveillance Service: Nowadays there is anincrease in the demands of mobile technology and surveillance with high and good quality of image capturing rapidly. Curvelets can be used to improve the image quality for surveillance with HDVC techniques and the novel discretecurvelet transform and modified WHOG for video surveillance and it is also for edge object detection too. This algorithm focuses on edge-based video compression, and it avoids the edge blurring and with reconstruction, with viable PSNR we can retain perpetual quality for surveillance services [16].

4.6 Medical Image Fusion: Nowadays, many sorts of medical images are produced in medical imaging modalities, which may aid doctors in diagnosing injuries and illnesses. Many researchers have worked on obtaining a fused medical picture that can provide extra concentration and image diagnosis based on data for medical evaluation.

In our methods, we apply GA to solve the suspicious and diffuse elements in the input image [17]. The genetic algorithm (GA) and the curvelets transform work together to generate a multimodal medical image.

4.7 Blind Image Watermaking. As we all know, obtaining the owner's mark from a watermarked image via a watermarking scheme does not help to save the original image. Nowadays, we only chose themes that are relevant to our day-to-day lives and may be tested, such as fraud detection, emergency clinics, and government divisions. They address the use of intelligent domain transforms such as the discrete shearlet transform and the discrete curvelet transform in this paper.

The secret image is embedded in the lowerband of the optimal coefficient with DCuT. The secret data was incorporated in the host image to keep it safe [18]. In the suggested watermarking approach, we evaluate 20 digital photos and apply various attacks to them.

4.8 Oscular Surface Vasculature Recognition. Vascular patterns on the white of the eyes are mostly found in the conjectival and episideral layers, and are referred to be oscular. The image produced with an OSV camera can also be used as a biometric recognition tool when combined with an image captured with a commercial RGB camera.

This research illustrates the capability of curvelettransform by extracting OSV features. In the curvelet domain, similar matrix and linear discriminants techniques are employed for matching.

The best error rate will be 0.2 percent if we use the multi-distance dataset for 50 volunteers with eye photographs obtained from 30, 150, and 250 cm using DSLR, and the error rate will increase if we use the second dataset. The experimental results back up the theoretically stated benefits of curvelettransform and its capacity to capture the data [19].

5 Conclusion and Future Scope

The main aim of this study is to know aboutcurvelets and to understand how curvelets are salutary to us. Things would become easier than beforedue to embracement ofcurvelets in real life.Curvelets can bring innovations in many fields i.e., the method of face recognition technique or how can we use vasculature recognition as a biometric, with much more security. The world is stepping towards digitization and a techno-friendly environment thenwork has to be done on the things which make the security and denoizing more accurate through which clear images can be observed.Curvelets are the most trending topics in the field of scientific computation and image processing and there are various fieldsin which curvelets have shown great contributions but the one thing on which the work is still on i.e., to have improved the directional elements and other promising abilities. But till now better results has been expected on it. But, yes there arevarious researches for this face recognition [21].

Another hot topic for curvelets is Automatic Speech Recognition (ASR), which uses a computer to convert audio speech into a word sequence. However, current research has some limitations, such as the inability to cope robustly with audio corruption caused by a variety of sources, such as engine noise, other people speaking noise, and environmental noise, or transmission of channel distortion. And, in the future, it's possible that the value of dynamic characteristics would be evident when testing in noisy rather than clean environments [22].

The beauty of curvelets over wavelets is that it can be constructed on general manifolds and there is very less work available in literature on general manifolds. Apart from it, curvelets have not been applied in the sense of adaptivity on general manifolds till now, so, we will explore the properties of curvelets to solve the complex differential equations in the future.

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