Stabilizing Constrained Control for Discrete-Time Multivariable Linear Systems via Positive Polyhedral Invariant Sets

Bourebia Ouassila

Automatic and Robotics Laboratory, Faculty of Sciences and Technology, Department of Electronics Constantine University, Constantine, Algeria

Corresponding author: Bourebia Ouassila, Email: bourbia_o@yahoo.fr, bourebia.ouassila@umc.edu.dz

In this work we propose a numerical method to compute stabilizing state feedback control laws and associated polyhedral invariant sets for multi-variable discrete systems. An MPC-based dual mode strategy and backward recursion from invariant set approach guarantees the feasibility and stability of the controller are investigated. The explicit solution subdivides the state space into regions, for each section, the set of admissible control laws is determined, the objective is to determine the region containing the admissible solutions starting the terminal region and backtracking. An illustrative example, showing the effectiveness of the proposed methods, is presented.

Keywords: Polyhedral, Invariant sets, Feasibility, Admissible solution, Dual mode.

2021. In Raju Pal & Praveen K. Shukla (eds.), *SCRS Conference Proceedings on Intelligent Systems*, 231–237. Computing & Intelligent Systems, SCRS, India. https://doi.org/10.52458/978-93-91842-08-6-22

1 Introduction

Positively invariant sets show an important role in the theory and applications of dynamical systems. Set invariance theory, very well probed in [1], it appears in many different problems to demonstrating stability such as in constrained control [2], with guaranteed invariance, stability and convergence properties [3]. There are many types of positively invariant sets such as polyhedral sets, ellipsoidal sets, Lorenz cones, etc. [3, 4]. We mainly consider convex polyhedral sets in this paper. The existence of an invariant set is equivalent to the presence of a Lyapunonv function and hence is equivalent to a stability test [5]. As we extend the class of system descriptions outside the class of linear systems, linear systems with constraints [6,7] are probably the most important class in practice. The most popular approaches or designing controllers for linear systems Model Predictive Control (MPC) [8] has become the accepted standard for complex constrained multivariable control problems in the process industries. A discussion on feasibility requires clear assumptions on the constraints in the optimization [9]. The controller is also given for the feasibility problem necessary and sufficient conditions for the existence of such controller are given [10]. To obtain a guarantee of robust stability in the presence of constraints, it is likely that the associated algorithm will give conservative performance and be valid only within a quite restricted region [5].

The objective is to determine the region containing the feasible admissible solutions of the problem MPC, the paper is organized as follows, first section introduces problem statement and proposed solution the second section gives an illustrative example, and a conclusion section closes the paper.

2 Problem Formulation

Consider the problem of regulating to the origin the discrete-time linear time invariant system represented by the following state equation such us:

$$x(k+1) = Ax(k) + Bu(k) \qquad (1)$$

where, $x(k) \in \mathbb{R}^n$ represents the state vector and $u(k)) \in \mathbb{R}^m$ stands for the control actions at time instant k., A and B are real $n \times n$ and n$\times$ m matrix respectively, i.e.$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

Supposed that exists around the origin a polyhedral positively invariant set $\mathcal{S}$ defined by the expression [2]

$$\mathcal{S}(G,\omega) = \{x \in \mathcal{R}^n / Gx \leq \omega\} \quad \text{With } \omega(i) > 0 \; fori = 1, \dots, g \qquad (2)$$

where, G is a matrix $G \in \mathbb{R}^{g\times n}$, and a vector $\omega \in \mathbb{R}^g$ with positive components, If $\mathcal{S}(G,\omega)$ is a convex positively invariant set of system, implies $x_k \epsilon \mathcal{S}$ and $x_{k+1} \epsilon \mathcal{S}$ [2].

It is emphasized that the MPC dual mode strategy makes use of invariant sets [8] and as such the algorithms are only defined when the state is within those sets the control low is defined:

$$U = -Fx\,. \qquad (3)$$

Suppose the control is subject to saturation constraints U_{min} and U_{max}

$$-U_{min} < U < +U_{max} \qquad (4)$$

Equation (2) implies that $x \in \mathcal{S}(G,\omega) \Rightarrow -\omega \leq Gx(k) \leq +\omega$

Substitute (1) in (2)

$$-\omega \leq G\big(Ax(k-1) + BU(k-1)\big) \leq +\omega \tag{5}$$

$$-\omega - GAx(k-1) \leq GBu(k-1) \leq +\omega - GAx(k-1) \tag{6}$$

$G \times B$ let's note C_1 matrix size $(g \times n) \times (n \times m) = (g \times m)$ using the left inverse pseudo matrix $C_1^+ = (C_1^{'} C_1)^{-1} C_1^{'}$ while $C_1^+ C_1 = I$

$$C_1^+ = ((GB)^{'}(GB))^{-1}(GB)^{'} \tag{7}$$

Substitute C_1^+ in (6) we find

$$C_1^+ (-\omega - GAx(k-1)) \leq u(k-1) \leq C_1^+(\omega - GAx(k-1)) \tag{8}$$

In addition, permissibility control region must respect saturation constraints. Consequently, the global Control constraints became:

$$\begin{cases} V_{min}^{k-1} \leq u(k-1) \leq V_{max}^{k-1} \\ U_{min} \leq u(k-1) \leq U_{max} \end{cases} \tag{9}$$

with

$$\begin{cases} V_{min}^{k-1} = C_1^+\big(-\omega - GAx(k-1)\big) \\ V_{max}^{k-1} = C_1^+(\omega - GAx(k-1)) \end{cases} \tag{10}$$

Feasible solutions occur if inequalities in (9) must satisfactory simultaneously. Set of permissible commands is then defined as thought intersection of these inequalities, which expressed by the resulting global contrast:

$$Max(U_{min}, V_{min}^{k-1}) \leq u(k-1) \leq \min(U_{max}, V_{max}^{k-1}) \tag{11}$$

The control law can be designed as

$$u(k-1) = \text{average}\ (Max\big(U_{min}, V_{min}^{k-1}\big), \min(U_{max}, V_{max}^{k-1})) \tag{12}$$

We can change average function in equation (12) by the Min or the Max function.

The states set for which the relation in (11) is fulfilled notedCF_{k-1}, if equation (11) is satisfied, there is at least one u_{k-1} transmitting $x_{k-1} \epsilon CF_{k-1}$ to $x_k \in \mathcal{S}$

from (11) x_{k-1} Is contained in the largest area C_{k-1} defined by:

$$V_{max}^{k-1} \geq U_{min} \,\&\, V_{min}^{k-1} \leq U_{max} \tag{13}$$

Substitute V_{min}^{k-1} and V_{max}^{k-1} we can have

$$C_1^+\omega - C_1^+ GAx(k-1) \geq U_{min} \tag{14}$$

$$-C_1^+\omega - C_1^+ GAx(k-1) \leq U_{max} \tag{15}$$

According to equations (13) and (14) by simple handling we find:

$$-(C_1^+\omega + U_{max}) \leq C_1^+ GAx(k-1) \leq C_1^+\omega - U_{min} \tag{16}$$

For step k-2

$$-(C_1^+\omega + U_{max}) \leq C_1^+ GA(Ax(k-2) + Bu(k-2)) \leq C_1^+\omega - U_{min} \tag{17}$$

$$-(C_1^+\omega + U_{max}) \leq C_1^+GA^2x(k-2) + C_1^+GABu(k-2)) \leq C_1^+\omega - U_{min} - (C_1^+\omega + U_{max}) - C_1^+GA^2x(k-2) \leq C_1^+GABu(k-2)) \leq -C_1^+GA^2x(k-2) + C_1^+\omega - U_{min} \quad (18)$$

Let's note$C_1^+GAB = C_2$

Knowing that matrix C_2 is a square matrix, we can use directly its inverse matrix C_2^{-1}, we obtain:

$$V_{min}^{k-2} \leq u(k-2) \leq V_{max}^{k-2} \quad (18)$$

with

$$\begin{cases} V_{min}^{k-2} = -C_2^{-1}[(C_1^+\omega + U_{max}) + C_1^+GA^2x(k-2)] \\ V_{max}^{k-2} = C_2^{-1}[-C_1^+GA^2x(k-2) + C_1^+\omega - U_{min}] \end{cases} \quad (19)$$

Given the saturation constraints defined by (4)

$$\begin{cases} V_{min}^{k-2} \leq u(k-2) \leq V_{max}^{k-2} \\ U_{min} \leq u(k-2) \leq U_{max} \end{cases} \quad (20)$$

Feasible solutions exist if inequalities in (21) must satisfactory simultaneously.

A set of permissible commands is then defined by intersection of these inequalities, which expressed by the following global inequality:

$$Max(U_{min}, V_{min}^{k-2}, V_{min}^{(k-1)}) \leq u(k-2) \leq \min(U_{max}, V_{max}^{k-2}, V_{max}^{(k-1)}) \quad (21)$$

The control low can be given by:

$$u(k-1) = \text{average}\ (Max(U_{min}, V_{min}^{k-2}, V_{min}^{(k-1)}), \min(U_{max}, V_{max}^{k-2}, V_{max}^{(k-1)})) \quad (22)$$

The set of states for which this relation is satisfiedCF_{k-2}, if this equation is satisfied, there is at least one u_{k-2} who transfer$x_{k-2} \epsilon CF_{k-2} \subset CF_{k-1}$ to $x_k \in \mathcal{S}$ from (22)

x_{k-2} Is contained in the largest area C_{k-2} defined by:

$$V_{max}^{k-2} \geq U_{min} \& \ V_{min}^{k-2} \leq U_{max} \quad (23)$$

with $V_{min}^{k-2}$and V_{max}^{k-2} are defined in equation (20) By simple manipulation

$$-C_2^{-1}(C_1^+\omega + U_{max}) - U_{max} \leq C_2^{-1}C_1^+GA^2x(k-2) \leq C_2^{-1}(C_1^+\omega - U_{min}) - U_{min} (24)$$

$$-C_2^{-1}(C_1^+\omega + U_{max}) - U_{max} \leq C_2^{-1}C_1^+GA^2(Ax(k-3) + Bu(k-3)) \leq C_2^{-1}(C_1^+\omega - U_{min}) - U_{min} \quad (26)$$

Let's note $C_2^{-1}C_1^+GA^2B = C_3$

Now suppose that in step $k-N$ the procedure will continue forN^{th}iteration, there is at least one admissible command u_{k-N}^f qui transmit$x_{k-N}$to$x_{k-N+1}$and such that:

$$\max(U_{min}, V_{min}^{(k-N)}, V_{min}^{(k-N-1)}, \ldots\ldots V_{min}^{(k-1)}) \leq u_{k-N}^f \leq \min(U_{max}, V_{max}^{(k-N)}, V_{max}^{(k-N-1)}, \ldots\ldots V_{max}^{(k-1)}) \quad (25)$$

where, $V_{min}^{(k-N)}$, $V_{min}^{(k-N)-1}$.... and $V_{max}^{(k-N)}, V_{max}^{(k-N-1)}, \ldots$.. are designed through the same steps established in the previous paragraphs.

The set of admissible control is defined by the simultaneous satisfaction of inequalities (27) this set of states for which the command exists $CF_{N-k-1}$with $CF_{N-k-1} \subset C_{N-k-1}$.

Finalizing proof of necessary condition. This isn't sufficient, there is no guarantee that acceptable solution will be found, this problem is general for predictive control [5]. However, with very large prediction horizons and simulation tests, it is possible, for a given problem, admissibility was guarantee.

Theorem

Necessary condition for a sequence of admissible control $u_j j = 0 \ldots N-1$ to transmit the current state x_0 insidethe invariant set such that:

$$\max\left(U_{min}, V_{min}^{(k-N)}, V_{min}^{(k-N-1)}, \ldots \ldots V_{min}^{(k-1)}\right) \leq u_{k-N}^{f} \leq \min\left(U_{max}, V_{max}^{(k-N)}, V_{max}^{(k-N-1)}, \ldots \ldots V_{max}^{(k-1)}\right)$$

where all its elements are calculated previously.

3 Simulation and Results

We consider the application of the feasible command obtained above to an example found in [4]. The system to be controlled is defined by:

$$x(k+1) = \begin{bmatrix} 0.7 & -0.1 & 0 & 0 \\ 0.2 & -0.5 & 0.1 & 0 \\ 0 & 0.1 & 0.1 & 0 \\ 0.5 & 0 & 0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0.1 \\ 0.1 & 1 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix} u(k) \tag{26}$$

$$\text{with} \qquad -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq U \leq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \forall k. \tag{27}$$

Polyhedral positively invariant set for this problem is defined

$$\Rightarrow -\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0.5 & 0 & 0.5 & 05 \\ -0.5 & 0 & -0.5 & -0.5 \end{bmatrix} x(k) \leq + \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \tag{28}$$

The stabilizing control under invariant set:

$$u = \left(-\begin{bmatrix} 0.0063 & -0.0045 & 0.0041 & 0.0017 \\ 0.0311 & -0.0577 & 0.0134 & 0.0020 \end{bmatrix}\right) x \tag{29}$$

The horizon is N = 20.

We can choose the control in the admissible area defined by (27). From this, three tests were carried out corresponding to three choices:

Test1: $u_f^j = \text{average}\ (\max\left(U_{min}, V_{min}^{(k-N)}, \ldots \ldots V_{min}^{(k-j)}\right), \min\left(U_{max}, V_{max}^{(k-N)}, . V_{max}^{(k-j)}\right))$

Test2: $u_f^j = \max\left(U_{min}, V_{min}^{(k-N)}, \ldots \ldots V_{min}^{(k-j)}\right)$

Test3 : $\min\left(U_{max}, V_{max}^{(k-N)}, . V_{max}^{(k-j)}\right)$

Figures Fig.1, Fig.2, and Fig.3 respectively give the states and the control signals for test n ° 1, the states in figure1, incur a large variation when entering in the invariant set.

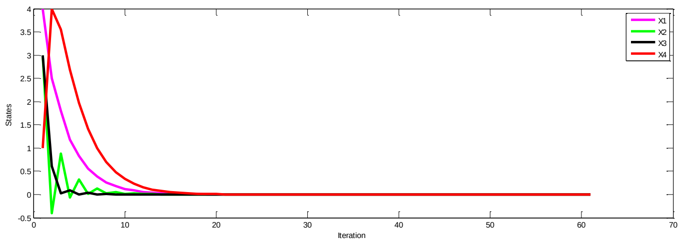

Fig. 1. The states

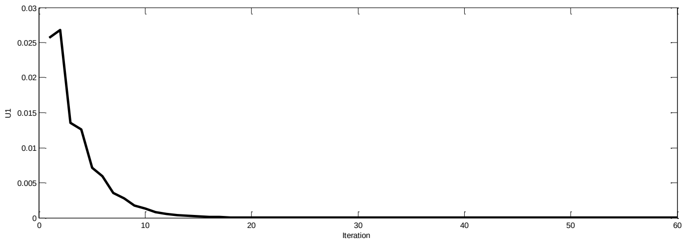

Fig. 2. Control law U1

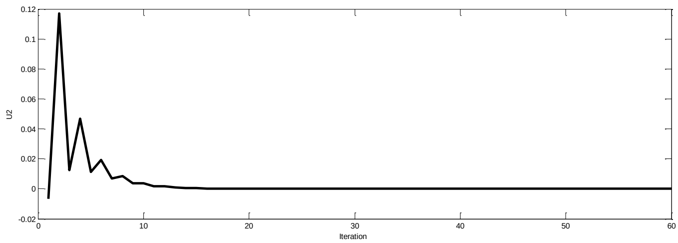

Fig. 3. Control law U2

4 Conclusion

In this work we have established an admissible plain solution introduced by the new methodology to solve the MPC dual mode control and optimization problem. Stability is guaranteed using the theory of invariant sets. The approach subdivides the state space into regions, for each region the set of admissible control laws is determined. The simulation results on the given problem are encouraging. The future work is to generalize the method for nonlinear and piecewise systems.

References

[1] Blanchini, F. (1999). Set invariance in control. *Automatica*, 35:1747-1767.

[2] George, B. (1988). On the positive invariant polyhedral sets for discrete-time linear systems. *International Journal of Control,* 47(6):1713-1726.

[3] Kuntsevich, V. M. and Pokotilo, V. G. (1995). Invariant Set Stability in Discrete Non-Linear Systems. *Journal of Applied Mathematics and Mechanics*. 58(5):815-823.

[4] Kerrigan, E. C. (2000). Robust Constraint Satisfaction: Invariant Sets and Predictive Control. Thesis.

[5] Rossiter, J. A. (2005). Model Based Predictive Control-A Practical Approach. Taylor & Francis e-Library.
[6] Michel, N. et al. (2018). Invariant Sets for Discrete-Time Constrained Linear Systems using a Sliding Mode Approach. *European Control Conference,* Cyprus.
[7] Sheer, S. and Gutman, P. (2016). A Novel Approach to the Computation of Polyhedral Invariant Sets for Constrained Systems. *IEEE Conference on Computer Aided Control System Design,* Argentina.
[8] Hairong, C. U. I. (2011). Offline Robust MPC based on Polyhedral Invariant Set. *Proceedings of the 30th Chinese Control Conference*, China.
[9] Kemgan, E. C. and Maciejowski, J. M. (2001). Robust feasibility in model predictive control: Necessary and sufficient conditions. *Proceeding of the 40th IEEE Conference on Decision and Control Orlando*, USA.
[10] Jingyi, L. et al. (2019). A Polytopic Invariant Set Based Iterative Learning Model Predictive Control. *IFAC*, 52(1):649–654.