Mathematical Analysis of Reaction Diffusion Equation using Laplace Transform in Life Sciences

Khagendra Singh[1], Vijay Vir[2], R.K. Srivastava[3]

Department of Applied Science (Mathematics), Dr. A. P. J. Abdul Kalam Technical University, Lucknow, U.P., India[1]

Department of Applied Science (Mathematics), Anand Engineering College, Agra, U.P., India[2]

Department of Mathematics, Agra College, Agra, U.P., India[2]

Corresponding author: Khagendra Singh, Email: khagendra01071980@gmail.com

This work is connected with the solution of reaction diffusion equation by Laplace transform in life sciences. Here we use the integral transform and inverse integral transform techniques to solve the boundary value problem with different boundary conditions. We also use some special function to solve the problem. The reaction diffusion equation and its solution is used in many fields like environment, medicines, oxygen diffusion through living tissues, problem of hemodailyser, spread of epidemics. These all areas are related to life sciences. To show its application in different areas, the examples are also illustrated.

Keywords: Reaction diffusion equation, partial differential equation, Laplace transforms, Integral transforms, life science

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1. Introduction

The development of reaction diffusion equation is not new one. In eighteenth century researchers used linear differential equations and their solution to find the solution of problems related with different fields of applied sciences and engineering. In next era, researchers started the linear and non-linear differential equations to solve the problems related with the same fields. After that few researchers started the application of integral transform methods to solve those problems which can be reduced in differential equation. Because of increasing demands for solution of problems related with physics, chemistry, mathematics and engineering problems, in Nineteenth century, researcher feels the need of some more general kind of the solution findings approaches that can help them to find the solution of problems in a systematic way. The area of solution related with partial differential equation is then introduced. Because of its easy applicability to the problems of different streams its popularity increases day by day. The partial differential equation with initial boundary conditions are started to solve with method of separation of variables and the process to solving these problems are also initiated with different methods. Different approaches are also tested to solve these problems but only few of them are successful. The area in which reaction diffusion equation is used is not scoped till mathematical applications, but it is extended diversely to many other area also. It is very much helpful in area of pharmacy, biology, physics, chemical reaction, environment and ecology related issues and many more. The literature review is given here in brief. In 1975 Aronson et al. [1] proposed non-linear diffusion in population genetics, combustion and nerve pulse propagation. In next decade 1986, Zhou [2] used differential transform and its application for electrical circuits. In late ninety's Grindord [3] explained the theory and applications of reaction-diffusion equations. In 2003, Oksendal [4] published stochastic differential equations. During 2008, Bataineh et al [5] focused on the Homotopy analysis method for Cauchy reaction diffusion problems. Kumar et al [6] explained the exact and numerical analysis of non-linear reaction diffusion equation by using the Coh Hopf transformation. In 2011 Du et al [7] gave spreading-Vanishing dichotomy in a diffusion logistic model with a free boundary. Bhadauria et al [8] explained a mathematical model to solve reaction diffusion equation using differential transform method. Kuttler [9] edited reaction diffusion equation with application. Du et al [10] focused on spreading and vanishing in non-linear diffusion problems with free boundaries. In 2019, Zhu [11] searched free boundary problems of several types of reaction diffusion systems. In 2021 Ran et al [12] proposed research on the reaction diffusion equation with Robin and free boundary condition. Assaely et al. [13] explained full finite element scheme for reaction diffusion systems on embedded curved surface in R3.

2. Mathematic Analysis of Problem

Reaction diffusion equation is:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + Ru \qquad (1)$$

With boundary conditions

$$u(0,t) = 0,\; u(x,0) = u_0 \qquad (2)$$

Where D is diffusion coefficient and R is reaction coefficient

Taking Laplace transform on both sides of (1)

$$L\left[\frac{\partial t}{\partial t}\right] = D\;L\left[\frac{\partial^2 u}{\partial x^2}\right] + R\;L[u]$$

$$\int_0^\infty \overline{e}^{st}\frac{\partial u}{\partial t}.dt = D\int_0^\infty \overline{e}^{st}\frac{\partial^2 u}{\partial x^2}dt + R\int_0^\infty \overline{e}^{st}u\,dt$$

$$\left[\overline{e}^{st}u(x,t)\right]_0^\infty + s\int_0^\infty \overline{e}^{st}u(x,t)\,dt = D\frac{d^2}{dx^2}\left\{\int_0^\infty \overline{e}^{st}u(x,t)\,dt\right\} + R\int_0^\infty \overline{e}^{st}u(x,t)\,dt \qquad (3)$$

Let $\int_0^\infty \overline{e}^{st} u(x,t)\,dt = \overline{U}$ (4)

Now (3) transforms into

$$0 - u(x,0) + s\overline{U} = D\frac{d^2\overline{U}}{dx^2} + R\overline{U}$$

$$D\frac{d^2\overline{U}}{dx^2} + (R-s)\overline{U} = -u_0$$ [From (2)]

$$\frac{d^2\overline{U}}{dx^2} + \frac{R-s}{D}\overline{U} = -\frac{u_0}{D}$$

Let $\frac{d}{dx} = D_1$

$$\left[D_1^2 + \frac{R-s}{D}\right]\overline{U} = \frac{-u_0}{D} \quad (5)$$

Auxiliary equation is $m^2 + \frac{R-s}{D} = 0$

Now there arise three cases

Case I: If R > s i.e. R – s < 0

Then $m = \pm\sqrt{\frac{s-R}{D}}$ (two different real roots)

$$\text{C.F.} = C_1 e^{\left(\sqrt{(s-R)/D}\right)x} + C_2 \overline{e}^{\left(\sqrt{(s-R)/D}\right)x} \quad (6)$$

$$\text{P.I.} = \frac{-u_0/D}{D_1^2 + \frac{R-s}{D}} = \frac{-u_0}{DD_1^2 + (R-s)} = f(x,s) \quad (7)$$

Solution is

$$\overline{U} = C_1 e^{\left(\sqrt{(s-R)/D}\right)x} + C_2 \overline{e}^{\left(\sqrt{(s-R)/D}\right)x} + f(x,s) \quad (8)$$

Case II: If R > s i.e. R – s > 0

Then $m = \pm i\sqrt{\frac{R-s}{D}}$ (we get two imaginary roots).

$$C.F. = C_3 \cos\left(\sqrt{\frac{R-s}{D}}\right)x + C_4 \sin\left(\sqrt{\frac{R-s}{D}}\right)x \quad (9)$$

$$P.I. = \frac{-u_0/D}{D_1^2 + \frac{R-s}{D}} = -\frac{u_0}{DD_1^2 + (R-s)} = f(x,s) \quad (10)$$

Solution is

$$\overline{U} = C_3 \cos\left(\sqrt{\frac{R-s}{D}}\right)x + C_4 \sin\left(\sqrt{\frac{R-s}{D}}\right)x + f(x,s)$$

(11)

Case III: When R = s

Then m = 0, 0 (we get two repeated real roots)

$$C.F. = C_1 + C_2 x \quad (12)$$

$$P.I. = \frac{-u_0 / D}{D_1^2} = g(x) \tag{13}$$

Solution is

$$\bar{U} = C_1 + C_2 x + g(x) \tag{14}$$

Taking inverse Laplace transform in (8), (11) & (14)

$$u(x,t) = \begin{cases} L^{-1}\left[C_1 e^{\left(\sqrt{(s-R)/D}\right)x} + C_2 e^{\left(\sqrt{(s-R)/D}\right)x} + L^{-1}\left[f(x,s)\right]\right], \; if \; R > s \\ L^{-1}\left[C_3 \cosh\sqrt{\frac{(s-R)}{D}}\, x + iC_4 \sinh\sqrt{\frac{(s-R)}{D}}\, x + L^{-1}\left[f(x,s)\right]\right], \; if \; R < s \\ (C_1 + C_2 x)\delta(t) + g(x)\delta(t), \; if \; R = s \end{cases}$$

$$u(x,t) = \begin{cases} e^{Rt}\left[\left[(C_1 + C_2)\right]\delta(t) - \frac{(C_1 - C_2)x}{2\sqrt{\pi}\sqrt{Dt}}\left(\frac{1}{t} - \frac{1}{9D}\left(\frac{x}{t}\right)^2 + \frac{1}{160D^2}\frac{x^4}{t^3} - \ldots\right) + F(x,t), \; if \; R < s\right] \\ e^{Rt}\left[C_1\delta(t) - \frac{iC_2}{2\sqrt{\pi}\sqrt{Dt}}\left(\frac{1}{t} - \frac{1}{9D}\left(\frac{x}{t}\right)^2 + \frac{1}{160D^2}\frac{x^4}{t^3} - \ldots\right)\right] + F(x,t), \; if \; R > s \\ \left[C_1 + C_2 x + g(x)\right]\delta(t), \; if \; R = s \end{cases} \tag{15}$$

$$\text{Where } F(x,t) = L^{-1}\left[f(x,s)\right] \tag{16}$$

Example: 1 the non-linear parabolic equation (1) and boundary condition is taken as $u_0 = 1$

$$u(x,t) = \begin{cases} e^{Rt}\left[(C_1 + C_2)\delta(t) - \frac{(C_1 - C_2)x}{2\sqrt{\pi}\sqrt{Dt}}\left(\frac{1}{t} - \frac{1}{9D}\left(\frac{x}{t}\right)^2 + \frac{1}{160D^2}\frac{x^4}{t^3} - \ldots\right) + 1\right], \; if \; R < s \\ e^{Rt}\left[C_1\delta(t) - \frac{iC_2 x}{2\sqrt{\pi Dt}}\left(\frac{1}{t} - \frac{1}{9D}\left(\frac{x}{t}\right)^2 + \frac{1}{160D^2}\frac{x^4}{t^3} - \ldots\right) + 1\right], \; if \; R > s \\ \left[C_1 + C_2 x - \frac{x^2}{2D}\right]\delta(t), \; if \; R = s \end{cases}$$

(17)

Taking u (0, t) = 0

$$0 = \begin{cases} (C_1 + C_2)\delta(t) + 1, & if \; R < s \\ C_1\delta(t) + 1, & if \; R > s \\ C_1\delta(t), & if \; R = s \end{cases} \tag{18}$$

$$u(x,t) = \begin{cases} \frac{a_0 x}{D^{1/2} t^{3/2}} e^{Rt}\left[1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right], \; if \; R < s \\ \frac{b_0 x}{D^{1/2} t^{3/2}} e^{Rt}\left[1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right], \; if \; R > s \\ \left(C_2 - \frac{x}{2D}\right)x\delta(t), \; \text{if R = s} \end{cases} \tag{19}$$

$$\text{Where } a_0 = \frac{(C_2 - C_1)}{2\sqrt{\pi}}, \; b_0 = \frac{-C_1}{2\sqrt{\pi}} \tag{20}$$

Example: 2 the non-linear parabolic equation (1) and boundary condition is taken as $u_0 = x$.

$$u(x,t) = \begin{cases} e^{Rt}\left[(C_1 + C_2)\delta(t) - \frac{(C_1 - C_2)x}{2\sqrt{\pi}\sqrt{D}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right) + x\right], \; if \; R < s \\ e^{Rt}\left[C_1\delta(t) - \frac{iC_2 x}{2\sqrt{\pi D}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2}\right) + x\right], \; if \; R > s \\ \left[C_1 + C_2 x - \frac{x^3}{6D}\right]\delta(t), \; if \; R = s \end{cases} \tag{21}$$

Taking u (0, t) = 0

$$0 = \begin{cases} (C_1 + C_2)\delta(t), & if\ R < s \\ C_1\delta(t), & if\ R > s \\ C_1\delta(t), & if\ R = s \end{cases} \tag{22}$$

$$u(x,t) = \begin{cases} \left[\frac{a_0 x}{\sqrt{D}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right) + x\right]e^{Rt}, if\ R < s \\ \left[\frac{b_0 x}{D^{1/2}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right) + x\right]e^{Rt}, if\ R > s \\ x\left(C_2 - \frac{x^2}{6D}\right)\delta(t), if\ R = s \end{cases} \tag{23}$$

Where $a_0 = \dfrac{C_2 - C_1}{2\sqrt{\pi}}\, b_0 = \dfrac{-iC_2}{2\sqrt{\pi}}$ (24)

Example: 3 the non-linear parabolic equation (1) and boundary condition is taken as $u_0 = \sin x$

$$u(x,t) = \begin{cases} e^{Rt}\left[(C_1 + C_2)\delta(t) - \frac{(C_1 - C_2)x}{2\sqrt{\pi}\sqrt{D}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right) + e^{-Dt}\sin x\right],\ if\ R < s \\ e^{Rt}\left[C_1\delta(t) - \frac{iC_2 x}{2\sqrt{\pi}\sqrt{D}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right) + e^{-Dt}\sin x\right],\ if\ R > s \\ \left(C_1 + C_2 x + \frac{1}{D}\sin x\right)\delta(t),\ if\ R = s \end{cases} \tag{25}$$

Taking u (0, t) = 0

$$0 = \begin{cases} (C_1 + C_2)\delta(t), & if\ R < s \\ C_1\delta(t), & if\ R > s \\ C_1\delta(t), & if\ R = s \end{cases} \tag{26}$$

$$u(x,t) = \begin{cases} \left[\frac{a_0 x}{\sqrt{D}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right) + e^{-Dt}\sin x\right]e^{Rt}, if\ R < s \\ \left[\frac{b_0 x}{\sqrt{D}t^{3/2}}\left(1 - \frac{1}{9D}\frac{x^2}{t} + \frac{1}{160D^2}\frac{x^4}{t^2} - \ldots\right) + e^{-Dt}\sin x\right]e^{Rt}, if\ R > s \\ \left(C_2 x + \frac{1}{D}\sin x\right)\delta(t), if\ R = s \end{cases} \tag{27}$$

Where $a_0 = \left(\dfrac{C_2 - C_1}{2\sqrt{\pi}}\right), b_0 = \dfrac{-iC_2}{2\sqrt{\pi}}$ (28)

3. Conclusion

Here we find the solution of reaction diffusion equation using Laplace transform. Reaction diffusion equation can be used in hemodialyser problem, in the problem of oxygen diffusion through living tissues, diffusion of immunoglobulin by lung tissues, and different kind of pollution problem; . It is widely used in area of medicines and epidemic spread analysis. Combining all the above aspects, we can conclude that the solution of reaction diffusion equation is useful in the area of life sciences. Scope of this paper is not limited to mathematical application only, but it can be used by user of different field such as pollution related problems, pandemic spread and controlling related problems and many more.

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